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Nonminimally coupled gravitational and electromagnetic fields: pp-wave solutions

Tekin Dereli*
Department of Physics, Koç University, 34450 Sarıyer, İstanbul, Turkey

Özcan Sert†
Department of Physics Engineering, İstanbul Technical University, 34469 Maslak, İstanbul, Turkey
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We give the Lagrangian formulation of a generic nonminimally extended Einstein-Maxwell theory with an action that is linear in the curvature and quadratic in the electromagnetic field. We derive the coupled field equations by a first-order variational principle using the method of Lagrange multipliers. We look for solutions describing plane-fronted Einstein-Maxwell waves with parallel rays. We give a family of exact pp-wave solutions associated with a partially massless spin-2 photon and a partially massive spin-2 graviton.

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I. INTRODUCTION

The predictions of the classical laws of electrodynamics have been verified to high levels of accuracy. These are the laws that are usually extrapolated to describe astrophysical phenomena under extreme conditions of temperature, pressure, and density. Any departures from these laws under such extreme conditions may be ascribed to new types of interactions between the electromagnetic fields and gravity. Here we consider nonminimal couplings of gravitational and electromagnetic fields described by a Lagrangian density that involves generic RF^2 terms. Such coupling terms were first considered by Prasanna [1]. They were soon extended and classified by Horndeski [2] to gain more insight into the relationship between space-time curvature and electric charge conservation. It is remarkable that a calculation in QED of the photon effective action from 1-loop vacuum polarization on a curved background [3] contributed similar nonminimal coupling terms. It was contemplated at about the same time that Kaluza-Klein reduction of a five-dimensional R^2 Lagrangian would induce similar nonminimal couplings in four dimensions [4]. A variation of an arbitrary Lagrangian with nonminimally coupled gravitational and electromagnetic fields in general may involve field equations of order higher than two. The nonminimal couplings in four dimensions classified by Horndeski are exactly those that involve at most second-order terms. These particular combinations are obtained by reduction of the Euler-Poincaré Lagrangian in five dimensions to four dimensions [5,6]. More recently, a 3-parameter family of nonminimally coupled Einstein-Maxwell field equations was considered in various aspects in a series of papers by Balakin and Lemos [7–9]. Intense gravitational fields that will be found near black holes behave as a specific kind of nonlinear medium in the presence of nonminimal couplings. Hence the electromagnetic waves that propagate in such media may imply new effects. Conversely, one should expect new gravitational effects induced by nonminimal couplings in the vicinity of neutron stars or magnetars where there are intense electromagnetic fields. Such new effects, if there are any, can be discussed in terms of exact solutions of the coupled field equations with appropriate isometries.

In this article, we formulate a nonminimally extended Einstein-Maxwell theory whose Lagrangian is linear in the curvature and quadratic in the electromagnetic field using the algebra of exterior differential forms. We derive the field equations by a first-order variational principle using the method of Lagrange multipliers. The resulting system of coupled equations we found is highly nonlinear. Exact solutions can be obtained in cases when the gravitational and electromagnetic fields have a high degree of symmetry. In particular, we consider solutions describing plane-fronted Einstein-Maxwell waves with parallel rays in Ehlers-Kundt form [10,11]. We present a family of exact solutions that are associated with a partially massless spin-2 photon and a partially massive spin-2 graviton.

II. NONMINIMALLY COUPLED EINSTEIN-MAXWELL FIELD EQUATIONS

We will derive our field equations by a variational principle from an action

$$I[e^a, \omega^a{}_b, F] = \int_M L = \int_M \mathcal{L}^*1, \qquad (1)$$

where $\{e^a\}$ and $\{\omega^a{}_b\}$ are the fundamental gravitational field variables and F is the electromagnetic field 2-form. The space-time metric $g = \eta_{ab} e^a \otimes e^b$ with signature $(-+++)$ and we fix the orientation by setting $*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$. Torsion 2-forms T^a and curvature 2-forms

*tdereli@ku.edu.tr
†sertoz@itu.edu.tr

$R^a{}_b$ of space-time are found from the Cartan-Maurer structure equations

$$de^a + \omega^a{}_b \wedge e^b = T^a, \tag{2}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b. \tag{3}$$

We consider the following Lagrangian density 4-form:

$$L = \frac{1}{2\kappa^2} R_{ab} \wedge^* (e^a \wedge e^b) - \frac{1}{2} F \wedge *F + \frac{\gamma}{2} R_{ab} \wedge F^{ab} * F, \tag{4}$$

where $\kappa^2 = 8\pi G$ is Newton's universal gravitational constant ($c = 1$) and γ is a coupling constant. The field equations are obtained by considering the independent variations of the action with respect to $\{e^a\}$, $\{\omega^a{}_b\}$ and $\{F\}$. The electromagnetic field components are read from the expansion $F = \frac{1}{2} F_{ab} e^a \wedge e^b$. We will be working with the unique metric-compatible, torsion-free Levi-Civita connection. We impose this choice of connection through constrained variations by the method of Lagrange multipliers. That is, we add to the above Lagrangian density the following constraint terms:

$$L_C = (de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a + dF \wedge \mu, \tag{5}$$

where λ_a's are Lagrange multiplier 2-forms whose variation imposes the zero-torsion constraint $T^a = 0$. We also use a first-order variational principle for the electromagnetic field 2-form F for which the homogeneous field equation $dF = 0$ is imposed by the variation of the Lagrange multiplier 2-form μ.

The infinitesimal variations of the total Lagrangian density $L + L_C$ (modulo a closed form) are found to be

$$\begin{aligned}\dot{L} + \dot{L}_C = {} & \frac{1}{2\kappa^2} \dot{e}^a \wedge R^{bc} \wedge *e_{abc} + \frac{1}{2} \dot{e}^a \wedge (\iota_a F \wedge *F - F \wedge \iota_a * F) + \dot{e}^a \wedge D\lambda_a - \frac{\gamma}{4} \dot{e}^a \wedge (\iota_a R_{bc} \wedge F^{bc} * F - R_{bc} \wedge F^{bc} \iota_a * F \\ & + F^{bc} \iota_a F \wedge *R_{bc} - F^{bc} F \wedge \iota_a * R_{bc}) + \dot{\lambda}_a \wedge T^a + \frac{1}{2} \dot{\omega}_{ab} \wedge (e^b \wedge \lambda^a - e^a \wedge \lambda^b) + \frac{\gamma}{2} \dot{\omega}_{ab} \wedge D(F^{ab} * F) - \dot{F} \wedge *F \\ & + \frac{\gamma}{2} \dot{F} \wedge F^{ab} * R_{ab} + \frac{\gamma}{2} F \wedge \dot{F}^{ab} * R_{ab} - \dot{F} \wedge d\mu,\end{aligned} \tag{6}$$

where a dot over a field variable denotes infinitesimal variations and we use shorthand notation $e^a \wedge e^b \wedge \cdots = e^{ab\cdots}$. Lagrange multiplier 2-forms λ_a are solved from the connection variation equations

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = \gamma D(F_{ab} * F). \tag{7}$$

It turns out that

$$\lambda_a = \gamma \iota^b D(F_{ba} * F) - \frac{\gamma}{4} e_a \wedge \iota^c \iota^b D(F_{bc} * F). \tag{8}$$

Thus the Einstein field equations are

$$\begin{aligned}-\frac{1}{2\kappa^2} R^{bc} \wedge *e_{abc} = {} & \frac{1}{2} (\iota_a F \wedge *F - F \wedge \iota_a * F) \\ & - \frac{\gamma}{4} (\iota_a R_{bc} \wedge F^{bc} * F - R_{bc} \wedge F^{bc} \iota_a * F + F^{bc} \iota_a F \wedge *R_{bc} - F^{bc} F \wedge \iota_a * R_{bc}) \\ & + \gamma F_{ac} \iota_b F \wedge *R^{cb} + \gamma D(\iota^b D(F_{ba} * F)) - \frac{\gamma}{4} e_a \wedge D(\iota^c \iota^b D(F_{bc} * F)),\end{aligned} \tag{9}$$

while the Maxwell equations read

$$dF = 0, \qquad d * (F - \gamma F_{ab} R^{ab}) = 0. \tag{10}$$

Electromagnetic constitutive equations

In general one may encode the effects of nonminimal couplings of the electromagnetic fields to gravity into the definition of a constitutive tensor. Maxwell's equations for an electromagnetic field F in an arbitrary medium can be written as

$$dF = 0, \qquad *d * G = J, \tag{11}$$

where G is called the excitation 2-form and J is the source electric current density 1-form. The effects of gravitation and electromagnetism on matter are described by G and J. To close this system we need electromagnetic constitutive relations relating G and J to F. Here we consider only the source-free interactions, so that $J = 0$. Then we take a simple linear constitutive relation

$$G = \mathcal{Z}(F), \tag{12}$$

where $\mathcal{Z}$ is a type-(2,2)-constitutive tensor. For the above theory we have

$$G = F - \gamma R_{ab} F^{ab}. \quad (13)$$

With these definitions, the nonminimally coupled Einstein-Maxwell action density simply becomes

$$L = \frac{1}{2\kappa^2} \mathcal{R} * 1 - \frac{1}{2} F \wedge *G. \quad (14)$$

The electric field e and magnetic induction field b associated with F are defined with respect to an arbitrary unit future-pointing timelike 4-velocity vector field U (''inertial observer'') by

$$e = \iota_U F, \qquad b = \iota_U * F. \quad (15)$$

Since $g(U, U) = -1$, we have

$$F = e \wedge \tilde{U} - *(b \wedge \tilde{U}). \quad (16)$$

Likewise the electric displacement field d and the magnetic field h associated with G are defined with respect to U as

$$d = \iota_U G, \qquad h = \iota_U * G. \quad (17)$$

Thus

$$G = d \wedge \tilde{U} - *(h \wedge \tilde{U}). \quad (18)$$

It is sometimes convenient to work in terms of polarization 1-form $p \equiv d - e = -\gamma(\iota_U R_{ab}) F^{ab}$ and magnetization 1-form $m \equiv b - h = \gamma(\iota_U * R_{ab}) F^{ab}$.

III. PLANE-FRONTED WAVE SOLUTIONS

We seek solutions that describe plane-fronted waves with parallel rays (pp waves). A generic pp-wave metric in Ehlers-Kundt form [10] is given by

$$g = 2dudv + dx^2 + dy^2 + 2H(u, x, y)du^2. \quad (19)$$

H is a smooth function to be determined. A convenient choice of orthonormal coframes is going to be used,

$$e^0 = \frac{H-1}{\sqrt{2}} du + dv, \qquad e^1 = dx,$$
$$e^2 = dy, \qquad e^3 = \frac{H+1}{\sqrt{2}} du + dv. \quad (20)$$

We may also exploit the advantages of complex coordinates in transverse plane by letting

$$g = 2dudv + 2dz d\bar{z} + 2H(u, z, \bar{z})du^2, \quad (21)$$

where

$$z = \frac{x + iy}{\sqrt{2}}, \qquad \bar{z} = \frac{x - iy}{\sqrt{2}}.$$

The nonvanishing Levi-Civita connection 1-forms are

$$\omega^0{}_1 = -\omega^1{}_3 = \frac{H_x}{2}(e^3 - e^0),$$
$$\omega^0{}_2 = -\omega^2{}_3 = \frac{H_y}{2}(e^3 - e^0). \quad (22)$$

Then the Einstein 3-forms $G_a = -\frac{1}{2} R^{bc} \wedge *e_{abc}$ become

$$G_0 = -G_3 = \frac{H_{xx} + H_{yy}}{2} * (e^3 - e^0), \qquad G_1 = 0 = G_2.$$

We consider an electromagnetic potential 1-form given as $A = a(u, x, y)du$ or $A = a(u, z, \bar{z})du$ for pp waves. Then

$$F = dA = a_x dx \wedge du + a_y dy \wedge du$$
$$= a_z dz \wedge du + a_{\bar{z}} d\bar{z} \wedge du. \quad (23)$$

We substitute these into (9) and (10) and after a lengthy calculation reach the final form of the nonminimally coupled Einstein-Maxwell equations,

$$H_{xx} + H_{yy} = -\kappa^2(a_x^2 + a_y^2) + \kappa^2\gamma((a_x^2)_{xx} + 2(a_x a_y)_{xy} + (a_y^2)_{yy}),$$
$$a_{xx} + a_{yy} = 0. \quad (24)$$

These equations can be rewritten in an invariant form on the transverse xy plane [11,12],

$$\Delta H = -\kappa^2 |\nabla a|^2 + \kappa^2 \gamma \mathrm{Hess}(a) - 2\kappa^2\gamma(\Delta(a\Delta a) - a\Delta(\Delta a) + (\Delta a)^2)$$
$$\Delta a = 0, \quad (25)$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the two-dimensional Laplacian,

$$|\nabla a|^2 = \left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2$$

is the norm-squared of the two-dimensional gradient, and

$$\mathrm{Hess}\,(a) = \begin{vmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{vmatrix} = a_{xx}a_{yy} - (a_{xy})^2$$

is the two-dimensional Hessian operator. In terms of complex coordinates, (25) simply reads

$$H_{z\bar{z}} = -\kappa^2 a_z a_{\bar{z}} + \kappa^2 \gamma a_{zz} a_{\bar{z}\bar{z}}, \qquad a_{z\bar{z}} = 0. \quad (26)$$

A nontrivial solution that depends on the coupling constant γ is obtained by letting

$$a(u, z, \bar{z}) = f_1(u)z + \bar{f}_1(u)\bar{z} + f_2(u)z^2 + \bar{f}_2(u)\bar{z}^2. \quad (27)$$

Then

$$\frac{1}{\kappa^2} H(u, z, \bar{z}) = f_3(u)z^2 + \bar{f}_3(u)\bar{z}^2 - |f_1(u)|^2|z|^2 - |f_2(u)|^2|z|^4 - f_1(u)\bar{f}_2(u)\bar{z}|z|^2 - f_2(u)\bar{f}_1(u)z|z|^2 + 4\gamma|f_2|^2|z|^2. \quad (28)$$

We note that the nonminimal coupling γ between the gravitational and electromagnetic waves is carried in the last term on the right-hand side of the expression above and affects only the space-time metric. Both the polarization

$p=0$ and the magnetization $m=0$ identically in the pp-wave geometry. We write

$$A=\mathcal{A}_{1+}+\mathcal{A}_{1-}+\mathcal{A}_{2+}+\mathcal{A}_{2-}, \tag{29}$$

where

$$\begin{aligned}\mathcal{A}_{1+}&=f_1(u)zdu=\bar{\mathcal{A}}_{1-},\\ \mathcal{A}_{2+}&=f_2(u)z^2du=\bar{\mathcal{A}}_{2-},\end{aligned} \tag{30}$$

and introduce $z=re^{i\theta}$ to show that

$$\begin{aligned}\mathcal{L}_{(1/i)(\partial/\partial\theta)}\mathcal{A}_{1\pm}&=\pm\mathcal{A}_{1\pm}\\ \mathcal{L}_{(1/i)(\partial/\partial\theta)}\mathcal{A}_{2\pm}&=\pm2\mathcal{A}_{2\pm}.\end{aligned} \tag{31}$$

$\mathcal{L}_X$ denotes the Lie derivative along the vector field X. Hence $\mathcal{A}_{1\pm}$, $\mathcal{A}_{1\pm}$ are null photon helicity eigentensors. Similarly, the metric tensor decomposes as

$$g=\eta+\mathcal{G}_0+\mathcal{G}_{1+}+\mathcal{G}_{1-}+\mathcal{G}_{2+}+\mathcal{G}_{2-}, \tag{32}$$

where η is the metric of Minkowski space-time and

$$\mathcal{G}_{1+}=-\bar{f}_1(u)f_2(u)z|z|^2du\otimes du=\bar{\mathcal{G}}_{1-}, \tag{33}$$

$$\mathcal{G}_{2+}=\bar{f}_3(u)z^2du\otimes du=\bar{\mathcal{G}}_{2-}, \tag{34}$$

$$\mathcal{G}_0=(-|f_1(u)|^2-|f_2(u)|^2|z|^4+4\gamma|f_2(u)|^2|z|^2)du\otimes du. \tag{35}$$

The $\mathcal{G}_{1\pm}$, $\mathcal{G}_{2\pm}$ are null g-wave helicity eigentensors for linearized gravitation about $\eta+\mathcal{G}_0$,

$$\mathcal{L}_{(1/i)(\partial/\partial\theta)}\mathcal{G}_{1\pm}=\pm\mathcal{G}_{1\pm}\qquad \mathcal{L}_{(1/i)(\partial/\partial\theta)}\mathcal{G}_{2\pm}=\pm2\mathcal{G}_{2\pm}. \tag{36}$$

This is a configuration associated with a partially massless spin-2 photon [13,14] and a partially massive spin-2 graviton.

IV. CONCLUSION

We have derived the field equations of a nonminimally coupled Einstein-Maxwell theory by a first-order variational principle using the method of Lagrange multipliers in the language of exterior differential forms. We give a class of exact, nontrivial pp-wave solutions. These solutions describe parallel propagating plane-fronted gravitational and electromagnetic waves that do not interact with each other in the Einstein-Maxwell theory. Here if only the standard degrees of polarization (±1 for the photon and ±2 for the graviton) are kept, no contribution arises from the nonminimal coupling constant γ. It is interesting to note, however, that if γ is kept it brings in ±2 polarization degrees of freedom for the photon together with ±1 polarization degrees of freedom for the graviton. The notion of a partially massless (spin-2) photon had been introduced before by Deser and Waldron [13,14]. On the other hand, the partially massive (spin-2) graviton here is new and it may find some observational evidence in the future. We wish to conclude by a few comments.

(i) It is possible to write an arbitrary linear combination of all RF^2-type invariants [2] as additional terms in the Lagrangian density 4-form

$$\begin{aligned}L'=&\frac{\gamma_2}{2}\iota^aF\wedge\mathcal{R}_a\wedge*F+\frac{\gamma_3}{2}\mathcal{R}F\wedge*F\\ &+\frac{\gamma_4}{2}R_{ab}F^{ab}\wedge F+\frac{\gamma_5}{2}\iota^aF\wedge\mathcal{R}_a\wedge F\\ &+\frac{\gamma_6}{2}\mathcal{R}F\wedge F,\end{aligned} \tag{37}$$

where $\mathcal{R}_a=\iota_bR^b{}_a$ are the Ricci 1-forms and $\mathcal{R}$ is the curvature scalar. Now the variational field equations become a lot more complicated but no essential insight is gained by such a generalization as far as the pp-wave solutions above are concerned.

(ii) A conformally scale invariant nonminimal coupling is achieved (for the case $\omega=-\frac{3}{2}$) by considering

$$\begin{aligned}L=&\frac{\phi}{2}R_{ab}\wedge*e^{ab}-\frac{\omega}{2\phi}d\phi\wedge*d\phi-\frac{1}{2}F\wedge*F\\ &+\frac{\gamma}{2\phi}C_{ab}\wedge F^{ab}*F,\end{aligned} \tag{38}$$

where ϕ is the dilaton and

$$C_{ab}=R_{ab}-\frac{1}{2}(e_a\wedge\mathcal{R}_b-e_b\wedge\mathcal{R}_a)+\frac{1}{6}\mathcal{R}e_{ab} \tag{39}$$

are the Weyl conformal curvature 2-forms ($\gamma_2=\gamma$, $\gamma_3=\frac{\gamma}{3}$, $\gamma_4=\gamma_5=\gamma_6=0$).

(iii) One may give up the zero-torsion constraint and vary the action (4) with respect to the metric and connection treated as independent variables. The equations resulting from the connection variations can be solved for the torsion 2-forms,

$$T_a=\kappa^2\gamma\iota^bD(\tilde{F}_{ab}*F)+\frac{1}{4}\kappa^2\gamma e_a\wedge\iota^c\iota^bD(\tilde{F}_{bc}*F). \tag{40}$$

Here we use the expansion $*F=\frac{1}{2}\tilde{F}_{ab}e^{ab}$. The exterior covariant derivatives on the right-hand side implicitly involve the contortion 1-forms. Therefore, the algebraic equations (40) admit a unique solution for the torsion 2-forms in terms of the tensor $\hat{D}(\tilde{F}_{ab}*F)$, but an explicit formula is not easy to obtain.

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